Tutorial 1: Complex Numbers and Functions

MATH2230B/C
Spring Semester 2017-18

A brief review:

(a) Complex number:

- $z = x + iy \in \mathbb{C}$, where $i = \sqrt{-1}$, $(x, y) \in \mathbb{R}^2$. (Modulus, conjugate, $\cdots$)
- Basic algebraic properties: commutative, associative, distributive.
- Polar coordinates: $z = re^{i\theta}$: geometric meaning of r and θ, Euler formulae. (Useful→ exercise: find roots)

(b) Geometric meaning

- Line: $\mathrm{Im}\Big((z - z_0)/z_1\Big) = 0$. (Meaning of > 0 and < 0?)
- Circle: $|z - z_0| = r$ or $\mathrm{Im}\left(\frac{z-z_1}{z-z_2}\Big/\frac{z_3-z_1}{z_3-z_2}\right) = 0$. (Orientation, $\cdots$)
- Extension: ellipse, hyperbola. (How about parabola?) → exercise below

(c) Some elementary transformations

- $z + z_0, e^{i\theta_0}z, |z_0|z$
- z^2
- Extension: e^z.

(d) Limits and continuity of functions

(e) Differentiability(Cauchy-Riemman equation) and analytic function

Pb.1 Determine the values of (a). $(2 + 2i)^{1/2}$ and (b). $1^{1/4}$.

Sol: (a).By Euler's formulae,

$$2 + 2i = \sqrt{2^2 + 2^2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{8}e^{\frac{\pi}{4}i + 2k\pi i}, \quad \forall k \in \mathbb{Z}.$$

Hence

$$(2 + 2i)^{1/2} = 8^{1/4}e^{\frac{\pi}{8}i + k\pi i}, \quad k \in \mathbb{Z}.$$

When $k = 0, 1$, these values are different.
Comment: You can also solve the quadratic $z^2 = z_0 \in \mathbb{C}$ by considering real and imaginary part.
(b). $1 = e^{2k\pi i}$ for $k \in \mathbb{Z}$.

□

Pb.2 Find the explicit form of ellipse, hyperbola and parabola in terms of the complex variable z.

Sol: (a). Ellipse: given any two focuses $z_1, z_2 \in \mathbb{C}$, and a real constant c_0,

$$|z - z_1| + |z - z_2| = c_0$$

(b). Hyperbola: similarly

$$|z - z_1| - |z - z_2| = \pm c_0.$$

(c). Parabola: given one point, say z_0, and a line passing through z_1 and z_2, then any point z on the parabola to z_0 has equal distance to the line passing through z_1, z_2. We denote the line by l, then

$$\mathrm{dist}(z, l) = |z_2 - z_1| \big| \mathrm{Im} \left[(z - z_1)/(z_2 - z_1) \right] \big|.$$

Hence a parabola can be represented by

$$|z_2 - z_1| \big| \mathrm{Im} \left[(z - z_1)/(z_2 - z_1) \right] \big| = |z - z_0|.$$

Comments: These are not the only representations for these curves. For instance, $x^2 - y^2 = c_0$ is a branch of hyperbola which can also be represented by $z^2 + \bar{z}^2 = 2c_0$. Here one can use

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}.$$

□

Pb.3 Exponential function as a mapping.
(a). Draw the range of the mapping $w = e^z$ with z on the vertical line $\{z : \mathrm{Re}(z) = x_0\}$.
(b). How about the strip $\mathrm{Re}(z) \in [1, 2]$?
(c). Furthermore, how about the a horizontal line and a general rectangle?

Sol: (a). Since $z = x_0 + iy$ for $y \in \mathbb{R}$, $w = e^{x_0 + iy} = e^{x_0} e^{iy} \in \partial \mathrm{D}_{r_0}$. Here D_{r_0} is the disk with center zero and radius $r_0 = e^{x_0}$; $\partial \mathrm{D}_{r_0}$ is the boundary of the disk described above, more explicitly, it is a circle with radius r_0.
(b). Annulus.
(c). A horizontal line can be expressed by $\mathrm{Im} z = y_0$, then $w = e^z = e^{x + iy_0} = e^x e^{iy_0}$ for $x \in \mathbb{R}$. This is a half line starting from the origin with angle determined by y_0.

□